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BRIEF COMMUNICATION

THE EQUATION OF MOTION OF A SMALL VISCOUS SPHERE IN AN UNSTEADY FLOW WITH INTERFACE SLIP

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INTRODUCTION

The equation for the creeping motion of a spherical particle was derived more than one hundred years ago as an exact solution of the Navier–Stokes equations in spherical co-ordinates. After Stokes presented the first such solution for the steady-state motion of a spherical particle, Boussinesq (1885) and then Basset (1888) solved the unsteady flow problem of a sphere in a spatially uniform ambient flow. Oseen (1927) introduced a higher order of accuracy to the equation of motion of a sphere. More recently Maxey & Riley (1983) derived the equation of motion of a rigid sphere in an unsteady and non-uniform flow without slip at the interface. Auton *et al.* (1988) derived expressions for the forces exerted on a spherical particle due to rotation. Most of the results on the equation of motion of spheres in viscous fluids are discussed in two recent monograms by Leal (1992) and Kim & Karila (1992) and papers by Yang & Leal (1991), Lovalenti & Brady (1993) and Galindo & Gerbeth (1993).

All the previous work for the forces on a sphere or the Lagrangian equations of motion of a sphere make use of the no-slip condition on the interface of the sphere and the surrounding fluid. There are several cases of practical interest, where tangential slip between the two phases has been observed. Interfacial slip has been advocated in the motion of small aerosol particles in the upper atmosphere. Also Leung & Crowe (1993) have used interfacial slip to analyze and explain the motion of nanocluster particles resulting from vapor condensation. These nanoclusters were part of a materials process, attracted by thermophoresis and collected on the surface of a metal. Given the presence of slip at the interface, the most plausible hypothesis about it is that the slip velocity is proportional to the tangential shear stress.

It is observed that in the case of a viscous sphere, there are two characteristic times ($\tau = \alpha^2/\nu$) for the fluid inside the sphere and the fluid outside the sphere. Therefore, there are two associated dimensionless parameters in the Laplace or Fourier domains ($\lambda^2 = s\alpha^2/\nu$). However (with the exception of the works by Galindo & Gerbeth and Lovalenti & Brady), we have found that in the studies pertaining to the motion of a viscous sphere in a fluid, the authors have effectively used the same dimensionless equation for the inside and the outside flow domains with only one time or length scale parameter. They have also used the equality of these dimensionless velocities in the boundary conditions of the equations. This method has led to incorrect results in the Fourier domain and the time domain. Correct ways of solving the problem are to use dimensional parameters throughout the solution, to modify the dimensionless governing equations, or to modify the no-slip boundary conditions by using the ratio of viscosities.

We have derived the correct solution for the equation of motion of a viscous sphere in the presence of interfacial slip, given the two characteristic times of the problem. The slip appears in the boundary conditions and is modeled by a relation similar to Coulomb's law of friction. It is observed that the inclusion of the two viscosities and the presence of slip affects both the steady-state drag term and the history term in the equation of motion of a sphere.

THE SPHERE IN AN UNSTEADY NON-UNIFORM VELOCITY FIELD

We consider the motion of a viscous fluid, which results in a velocity field $u_i(x_j, t)$ as measured with respect to a stationary co-ordinate system $0x_1x_2x_3$. We also consider the motion of a sphere of radius α whose center is located at $Y_i(t)$ and is moving with velocity $V_i(t)$ with respect to the same co-ordinate system. The presence and motion of the sphere disturbs the velocity field $u_i(x_j, t)$ and creates a new flow field, which is denoted as $q_i(x_j, t)$. In order to study the motion of the sphere it is convenient to change the co-ordinate system to one located at the center of the sphere and moving with it. Thus, a change of variables is made for the moving frame of reference:

$$\mathbf{z} = \mathbf{x} - \mathbf{Y}(t) \qquad [1]$$

Hence, a relative velocity of the fluid with respect to the moving sphere may be derived in the new system of co-ordinates:

$$\mathbf{w}(\mathbf{z}, t) = \mathbf{q}(\mathbf{x}, t) - \mathbf{V}(t) \qquad [2]$$

The continuity and momentum equations for the fluid in the $0z_1z_2z_3$ system may be written as follows:

$$\frac{\partial w_i}{\partial z_i} = 0 \qquad [3a]$$

and

$$\rho\left(\frac{\partial w_i}{\partial t} + w_j\frac{\partial w_i}{\partial z_j}\right) = \rho\left(g_i - \frac{\mathrm{d}V_i}{\mathrm{d}t}\right) - \frac{\partial p}{\partial z_i} + \mu\frac{\partial^2 w_i}{\partial z_j \partial z_j} \qquad [3b]$$

where ρ and μ are the fluid density and dynamic viscosity and g_i is the acceleration due to gravity.

Any slip condition at the interface must be reflected on the boundary conditions of the differential equations. In the radial direction there can be no slip, since this would violate an essential kinematic condition. Therefore, slip may only be present in the tangential direction. Given the presence of slip, the most plausible assumption for a closure equation is that the tangential velocity difference is proportional to the tangential shear stress $\sigma_{r\theta}$ as alluded to by Basset (1888). This postulate is analogous to Coulomb's law of friction for solid surfaces moving relatively to each other. The coefficient of kinetic friction, β, should depend on the material properties of the fluid and the sphere and not on the characteristics of the velocity field developed.

Hence, the boundary conditions for [3] at the surface of the sphere may be written in the following form:

$$\delta w_\theta = \frac{\sigma_{r\theta}}{\beta} \quad \text{and} \quad \delta w_r = 0 \quad at\,|\mathbf{z}| = \alpha. \qquad [4]$$

It must be pointed out that $\beta = \infty$ corresponds to the zero slip condition, while $\beta = 0$, corresponds to the perfect slip case. The boundary condition at the far field is that the effect of the sphere on the flow velocity diminishes to zero.

Following the assumptions and method used by others (e.g. Maxey & Riley) we have separated the flow field outside the sphere in the undisturbed flow field w_i^0 and the disturbance caused by the presence of the sphere w_i^1. The two resultant flow fields satisfy the condition $w_i^0 + w_i^1 = w_i$ at all points. Furthermore, the decomposition of the velocity fields satisfy their corresponding momentum equations. Thus, one may obtain the total force due to the undisturbed velocity field and, hence, derive the lagrangian equation of motion of the sphere. The latter is given as follows in the original frame of reference $0x_1x_2x_3$:

$$m_s\frac{\mathrm{d}V_i}{\mathrm{d}t} = (m_s - m_f)g_i + m_f\left.\frac{\mathrm{D}u_i}{\mathrm{D}t}\right|_{Y(t)} + F_i^1. \qquad [5]$$

Here m_s is the mass of the sphere, m_f is the mass of the fluid, which occupies the same volume as the sphere and F_i^1 the force exerted by the flow field due to the disturbance. The derivative $\mathrm{D}/\mathrm{D}t$ is the lagrangian derivative following a fluid element. It is apparent that for the calculation of the equation of motion of the sphere the force F_i^1 must be determined.

F_i^1 is first determined in the Laplace domain and then transformed analytically or numerically in the time domain. Since the expression for this disturbance force has been derived before (in the non-slip case) by several methods and the derivation is lengthy, we will not include it in this brief paper. This force in the Laplace domain and in the $0x_1x_2x_3$ system is as follows:

$$\overline{F_i^1} = -6\pi\alpha\mu_1[\bar{V}_i(s) - u_i(\mathbf{Y}(t), s)]$$
$$\times\left\{\frac{\lambda_1^2}{9} + \lambda_1 + 1 - \frac{(\lambda_1+1)^2\langle[\lambda_2^3 - \lambda_2^2\tanh(\lambda_2) - 2f(\lambda_2)]\kappa\sigma + f(\lambda_2)\rangle}{[1+\sigma(\lambda_1+3)][\lambda_2^3 - \lambda_2^2\tanh(\lambda_2) - 2f(\lambda_2)]\kappa + (\lambda_1+3)f(\lambda_2)}\right\}. \quad [6]$$

where the overbar denotes the Laplace transform of a function, κ is the ratio of the fluid to sphere dynamic viscosities ($\kappa = \mu_1/\mu_2$) and σ is a dimensionless parameter related to the slip coefficient ($\sigma = \mu_1/\beta\alpha$). The parameters λ_1 and λ_2 are two dimensionless length scales for the fluid and for the sphere, defined as follows:

$$\lambda_1 = \sqrt{\frac{s\alpha^2}{v_1}} \quad \text{and} \quad \lambda_2 = \sqrt{\frac{s\alpha^2}{v_2}} \quad [7a]$$

with s being the Laplace transformation variable and v the kinematic viscosity. The function f is defined by the following expression:

$$f(\xi) = (\xi^2 + 3)\tanh(\xi) - 3\xi. \quad [7b]$$

Because of its complexity, [6] cannot be transformed analytically to yield the resulting disturbance force in the time domain. However, parts of it may be transformed. Thus, the first term in the braces will yield an "added mass" contribution, the second term (which is proportional to the square root of s) will yield a history integral and the third term the steady-state drag term (as all three are known in the case of the solid sphere). Given the complexity of the last term, it is difficult to speculate the type of its contribution on the resultant force. The last term may be considered as a residue of all the above terms, applicable to the non-rigid spheres. In some special cases it becomes a "new memory integral," as was called by Lawrence Weinbaum (1986), but in general its contribution is more complex than a memory term.

It is possible, under certain circumstances to transform the last term analytically and, hence, to derive the complete equation of motion of a viscous sphere for some special cases.

SPECIAL CASES

No-slip ($\sigma = 0$)

This is probably the most widely used case with droplet flows. In this case [6] becomes:

$$\overline{F_i^1} = -6\pi\alpha\mu_1[\bar{V}_i(s) - \bar{u}_i(\mathbf{Y}(t), s)]\left\langle\frac{\lambda_1^2}{9} + (\lambda_1 + 1) - \frac{(\lambda_1+1)^2 f(\lambda_2)}{[\lambda_2^3 - \lambda_2^2\,\mathrm{tahn}(\lambda_2) - 2f(\lambda_2)]\kappa + (\lambda_1+3)f(\lambda_2)}\right\rangle. \quad [8]$$

This expression is the same as the one derived by Galindo & Gerbeth (1993) and by Lovalenti & Brady (1993). It is still impossible to transform it into the time domain analytically. It must be pointed out, however, that only if $\lambda_1 = \lambda_2$, it is essentially the same equation as the one derived by Kim & Karila (1991) and Yang & Leal (1991).

Asymptotic behavior without slip. It is of interest to study the short- and long-term behavior of the history force on the sphere. A glance at [8] proves that the first three terms of the parentheses on the right-hand side will yield the added mass term, the history integral for the rigid sphere and the steady-state drag contribution. The rest may be considered as a residue function $L(\lambda_1, \lambda_2, \kappa)$. In this case the force due to the disturbance field is given by the following equation:

$$\overline{F_i^1} = -6\pi\mu_1\alpha\bar{H}_i\left[\frac{2+3\kappa}{3(1+\kappa)} + \frac{\kappa}{1+\kappa}\lambda_1 + \frac{1}{9}\lambda_1^2 + L(\kappa, \lambda_1, \lambda_2)\right], \quad [9]$$

where the vector $H_i(t)$ is defined as:

$$H_i(t) = V_i(t) - u_i(\mathbf{Y}(t), t).$$

The residue function $L(\lambda_1, \lambda_2, \kappa)$ is written in terms of one of the λs and the ratio of the kinematic viscosities γ. Taking the limit for short times (very high s) one obtains the following expression for the residue function:

$$L(\lambda_1, \gamma, \kappa) = \frac{\kappa(\gamma - 1)\lambda_1}{(\gamma\kappa + 1)(1 + \kappa)} + \frac{\gamma(\gamma - 6)\kappa^2 - (4\gamma - 3)\kappa + 4}{3(1 + \kappa)(\gamma\kappa + 1)^2} - \frac{1}{\lambda_1}\frac{(\gamma^3 - 3\gamma)\kappa^2 - (3 + 4\gamma^2)\kappa + 4\gamma}{\gamma(\gamma\kappa + 1)^3} + O(\lambda_1^{-2}). \quad [10]$$

The last expression may be transformed into the time domain to yield the following expression for the residue term at the limit of short times:

$$F_{Mi} = 6\pi\alpha\mu_1 \int_0^t G(t - \tau, \kappa, \gamma)\frac{dH_i(\tau)}{d\tau}\, d\tau, \quad [11a]$$

where

$$G(t, \kappa, \gamma) = \frac{\kappa(\gamma - 1)}{(1 + \kappa)\sqrt{\pi t}} - \frac{5\gamma\kappa - 4}{3(1 + \kappa)(\gamma\kappa + 1)} + \frac{(3\gamma - \gamma^3)\kappa^2 + (3 + 4\gamma^2)\kappa - 4\gamma}{\gamma(\gamma\kappa + 1)^3}\sqrt{\frac{t}{\pi}}. \quad [11b]$$

Similarly one may obtain an analytical expression for the residue term at long times, by letting s approach 0. The resulting expression for the predominant term in the function L is as follows in this asymptotic limit:

$$L(\lambda_1, \kappa, \gamma) = \frac{4 + 3\kappa}{9(1 + \kappa)^2}\lambda_1 + O(\lambda_1^2) \quad [12]$$

The last expression may be transformed analytically in the time-domain to yield the following expression, which is actually a memory integral term:

$$F_{Mi} = 6\pi\alpha\mu_1 \int_0^t \frac{4 + 3\kappa}{9(1 + \kappa)^2\sqrt{\pi(t - \tau)}}\frac{dH_i}{d\tau}\, d\tau. \quad [13]$$

It is observed that, at high values of t, the residue term is independent of the ratio γ. Since the equations are written in terms of λ_1, this means that the new term is independent of the dimensionless time-scale λ_2. This may be interpreted in the following way: all memory integral terms represent the effect of the diffusion of vortices inside a fluid domain. Given the small size of the sphere, at long times the vortices have already been "diffused" inside the fluid sphere, and, hence, their effect on the interior domain dissipates to zero. On the contrary the domain of the outside fluid is much larger and the created vortices continue to "diffuse" even at longer times, thus affecting the current state of the sphere.

A solid sphere ($\kappa = \infty$)

In this case the viscosity of the sphere is very high in comparison to that of the fluid. Thus, there is only one dimensionless number λ for the diffusion of momentum, outside the sphere. The force expression as given in [6] may be transformed into the time domain to yield the following expression:

$$F_i^1(t) = -\tfrac{1}{2}m_f\frac{dH_i(t)}{dt} - \frac{1 + 2\sigma}{1 + 3\sigma}6\pi\alpha\mu_1 H_i(t) - \frac{[1 + 2\sigma]^2}{\sigma[1 + 3\sigma]}6\pi\alpha\mu_1 \times \int_0^t \exp\left(\frac{[1 + 3\sigma]^2\nu_1[t - \tau]}{\alpha^2\sigma^2}\right)\mathrm{Erfc}\left(\frac{1 + 3\sigma}{\alpha\sigma}\sqrt{\nu_1[t - \tau]}\right)\frac{dH_i(\tau)}{d\tau}\, d\tau. \quad [14]$$

In the case of zero slip ($\sigma = 0$) [14] yields the known expression for a rigid sphere, which was first derived by Boussinesq (1885) and Basset (1888).

In this case one may combine [5] and [6] to derive an analytical expression of the equation of motion of the solid sphere, which is as follows:

$$m_s \frac{dV_i}{dt} = [m_s - m_f]g_i + m_f \frac{Du_i}{Dt}\bigg|_{Y(t)} - \tfrac{1}{2}m_f \frac{dH_i(t)}{dt} - \frac{1+2\sigma}{1+3\sigma} 6\pi\alpha\mu_1 H_i(t)$$

$$- \frac{[1+2\sigma]^2}{\sigma[1+3\sigma]} 6\pi\alpha\mu_1 \int_0^t \exp\left(\frac{[1+3\sigma]^2 \nu_1 [t-\tau]}{\alpha^2\sigma^2}\right) \mathrm{Erfc}\left(\frac{1+3\sigma}{\alpha\sigma}\sqrt{\nu_1[t-\tau]}\right)\frac{dH_i(\tau)}{d\tau}\, d\tau. \quad [15]$$

For clarity, in [14] and [15] parentheses enclose arguments of functions, while multiplications are denoted by the square brackets; exp denotes the exponential function; Erfc denotes the complementary error function.

Regarding the physical interpretation of the various terms, the left-hand side of [15] is the mass times acceleration term on the sphere in a general frame of reference. The first term in the right-hand side is the gravity/bouyancy force acting on the sphere. The second term is the contribution of the acceleration of the fluid to the motion of the sphere. The third term is often called the added mass force. The fourth term is the usual steady-state drag force term. The fifth term on the right-hand side is the history force acting on the particle. In the non-slip case this term is often referred to as the "Basset term" despite the fact that Boussinesq derived it and published about it three years before Basset (Vojir & Michaelides 1993). It is apparent that if slip is present on the interface the kernel of the history integral is remarkably different than the one derived by Boussinesq and Basset for a rigid sphere.

It is of interest to examine the effect σ has on the magnitude of the history term. For this reason calculations were made for the simple case where the particle follows a sinusoidal motion in a stagnant fluid. For the calculations the fluid velocity was assumed to be zero and the particle velocity to be given by the expression $V_i(t) = (0, 0, \cos \omega_0 t)$ with a dimensionless frequency ω_0 equal to 20. Figure 1 depicts the results for the dimensionless history term as a function of time, with σ a parameter. It is observed that the effect of the history terms becomes highest when there is no slip or very little slip on the interface. When the slip parameter σ increases, the magnitude of the history term decreases. Very little difference was observed in the calculations on the history term for values of σ greater than five.

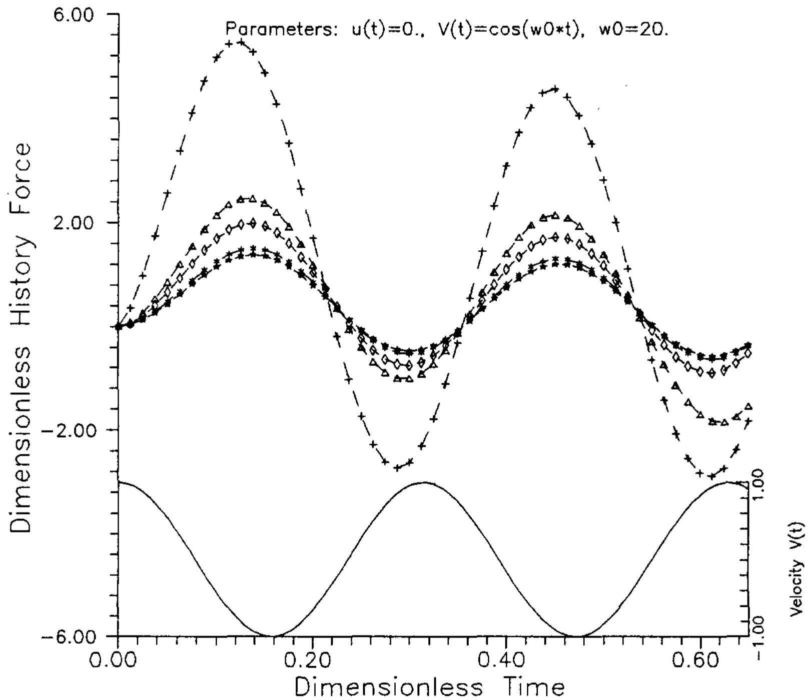

Figure 1. The effect of the dimensionless slip parameter σ on the magnitude of the history term. $+$, $\sigma = 0$; Δ, $\sigma = 0.5$; $\Diamond$, $\sigma = 1.0$; ✱, $\sigma = 5.0$; and ✡, $\sigma = \infty$.

A sphere with perfect slip ($\sigma = \infty$)

In this case the force due to the disturbance flow in the Laplace domain is:

$$\overline{F_i^{\rm l}} = -6\pi\mu_1\alpha\left[\frac{\lambda_1^2}{9} + \frac{2(1+\lambda_1)}{3+\lambda_1}\right][\overline{V_i}(s) - \overline{u_i}(\mathbf{Y}_i(t), s)]. \qquad [16]$$

An analytical expression for the total force in the time domain may be obtained. This expression yields the following equation of motion for the sphere:

$$m_s\frac{\mathrm{d}V_i}{\mathrm{d}t} = [m_s - m_f]g_i + m_f\left.\frac{\mathrm{D}u_i}{\mathrm{D}t}\right|_{\mathbf{Y}(t)} - \tfrac{1}{2}m_f\frac{dH_i(t)}{\mathrm{d}t} - 4\pi\alpha\mu_1 H_i(t)$$
$$- 8\pi\alpha\mu_1\int_0^t \exp\left(\frac{9\nu_1[t-\tau]}{\alpha^2}\right)\mathrm{Erfc}\left(\frac{3\sqrt{\nu_1[t-\tau]}}{\alpha}\right)\frac{\mathrm{d}H_i(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau. \qquad [17]$$

This expression is fundamentally the same as the one derived by Morrison & Stewart (1976) who considered the motion of an inviscid bubble. They too derived an expression for the total force on a small bubble in an unsteady flow field, under the assumption $\mu_2 = 0$. Their expression gives the resultant force in an implicit way through an integrodifferential expression, while [17] is given in an explicit form. It must be pointed out that [16] is identical to the corresponding expression derived by Morrison & Stewart. Equation [17] also yields the correct expression for the steady-state drag on a small sphere with perfect slip, which is $F_{\mathrm{D}i} = 4\pi\mu\alpha H_i$, which also appears in Happel & Brenner (1986). From the two analyses it is concluded that the assumptions of perfect slip or inviscid sphere result in the same final expression for the equation of motion of the sphere.

CONCLUSIONS

The equation for the creeping motion of a viscous sphere in an unsteady velocity field has been derived for the general case, where finite slip is present at the interface. A careful solution of the conservation equations reveals that the problem has two length scales for the diffusion of momentum (inside and outside the sphere). Both of these length scales come into the expression of the force due to the disturbance flow field. An analytical expression for the total force may be obtained in the Laplace space. This expression cannot be transformed in the time domain analytically. It may be transformed in the case of a rigid sphere and an inviscid bubble. In these cases it was observed that the history integral becomes more complicated than the one originally derived by Basset & Boussinesq. Calculations for the simple case of the sinusoidal motion of a particle in a still fluid have shown that the slip on the sphere diminishes the effect of the history term in the determination of the instantaneous velocity of the sphere.

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